Least Squares Phase Estimation of Mixed Signals
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Abstract
Estimating the phase of sinusoids in noise in contrast to amplitude and frequency components has been less addressed in previous studies. In this paper, we derive the least squares phase estimator (LSPE) solution to recover the phase of an underlying signal observed in noise. Through Monte-Carlo simulations, we demonstrate the robustness of the proposed phase estimator against the modeling error. The proposed phase estimator is further evaluated in the speech enhancement setup to assess how much improvement is obtained by replacing the noisy phase with the LSPE when reconstructing the enhanced speech signal. Significant improvement in speech quality and speech intelligibility is obtained by replacing the noisy phase with the estimated phase provided by the proposed LSPE once the ambiguity in the phase candidates is removed.

Index Terms: Least squares, phase estimation, speech enhancement, speech quality.

1. Introduction
In estimation theory and signal detection, the ultimate goal is to find optimal estimators to recover a sinusoid observed in noise. In particular, we are interested to find estimates for the triple parameters: amplitude, frequency and phase for sinusoidal components of a signal of interest [1]. Such a general problem definition holds for a wide range of signal and audio processing applications, to name a few: signal detection [2], speech enhancement [3] and radar signal processing [4].

While there are many previous studies focused on deriving estimators of the parameters of sinusoids observed in noise [1,5], the issue of estimating the phase of the signal in noise has often been neglected. For example, in speech enhancement, the phase spectrum has been considered to be unimportant [6,7] as the noisy phase was shown to provide the optimal estimate for clean phase for the signal components of high signal to noise ratio (SNR) [8,9]. This optimality is only valid under the independence assumption of the STFT signal coefficients which is not always true in practice. Therefore, the choice of noisy phase limits the achievable performance by the existing signal enhancement methods. In fact, the importance of phase spectrum has been reported with positive results in the speech quality [10–18], speech intelligibility [19], speech recognition [20], and other signal processing applications [7] (see [21] for an overview). These findings motivate us to find estimators for the phase of the sinusoidal components observed in noise.

In this paper, we propose a least squares (LS) solution for estimating the phase of the underlying signals from a mixed signal. The derivations presented here require no prior assumption on the distribution on the underlying signals, in contrast to the conventional estimators where spectral amplitude and phase are assumed to be independent. By means of Monte-Carlo simulations on the synthetic sinusoids in noise, the proposed estimator is evaluated in terms of the mean square error (MSE) and an unwrapped phase distortion metric proposed in [22]. This estimator is eventually tested on real speech signals in the context of speech enhancement and is shown to improve the perceived quality as well as speech intelligibility of the signals using noisy phase for different noise scenarios and signal-to-noise ratios.

2. Problem formulation
Consider a single channel mixed signal can be modeled, using a physical interpretation, as the sum of two signals coming from different sources (positioned at $r_1$ and $r_2$, respectively). The received mixed signal is given by

$$ y(t) = s_1(r_1, t) + s_2(r_2, t). \quad (1) $$

Taking the Fourier transform with respect to the time $t$ from Eq. (1), we obtain

$$ Y^e(\omega) = S_1^e(\omega_1) + S_2^e(\omega_2), \quad (2) $$

where $Y^e, S_1^e, S_2^e \in \mathbb{C}$, can be represented in a polar form as follows:

$$ Y(\omega)e^{j\phi_e(\omega)} = S_1(\omega)e^{j\phi_1(\omega)} + S_2(\omega)e^{j\phi_2(\omega)}, \quad (3) $$

where $\phi_1$ and $\phi_2$ denote the phase of the mixed signal and the underlying $i$th signal in the mixture with $i \in \{1, 2\}$ while $S_i(\omega)$ represents the $i$th signal magnitude with $Y(\omega)$ as the mixture magnitude spectrum at frequency $\omega$. Throughout this paper, for simplicity we will use $Y(\omega) = Y, S_i(\omega) = S_i$ and $\phi_i(\omega) = \phi_i$. The signal + noise interaction in Eq. (3) can be geometrically interpreted as shown in Figure 1, where $Y^e$ is the sum of vectors $S_1^e$ and $S_2^e$ in the complex plane.

We are interested in computing the phase of $S_i$ given the noisy phase $\phi_e$ and good estimations of the magnitude spectra $S_i$ and mixed signal observations $Y$ and $\phi_e$. In this paper, we derive the Least Squares Phase Estimator (LSPE) only for $\phi_1$. Without loss of generality, an extension for the proposed LSPE is straightforward for the second source phase ($\phi_2$). The problem can be stated as finding the phase that minimizes the difference between $Y^e$ and the sum of signals $S_1^e$ and $S_2^e$ in the least squares sense, given by

$$ d_{SPE}(\phi_1) = |Y^e - (S_1^e + S_2^e)|^2. \quad (4) $$

The derivations presented here require no prior assumption

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Using the method of Lagrangian multipliers, the mixed signal in the Fourier domain can be written as a derivation of the two sets. Using the phase difference, the observations in the least squares sense (see also [23] for a detailed discussion). Therefore, there is ambiguity in the set of phase candidates.

Nevertheless, as noted in [22], an additional optimality criterion on the underlying phase can be incorporated via combining the cost function Eq. (4) with an unwrapped cosine distortion as:

\[ d_{\cos}(\phi_1, \hat{\phi}_1) = 1 - \cos(\phi_1 - \hat{\phi}_1). \] (5)

The distortion metric, in contrast to the mean square error (MSE) criterion, is invariant under modulo 2π transformation of the estimation error. For small estimation errors, it resembles the squared error distortion criterion. By doing this, the resulting estimate does not affect the estimation of the spectral magnitude [22]. Using the method of Lagrangian multipliers, this constrained problem can be formulated as minimizing the Lagrangian function:

\[ \Lambda(\hat{\phi}_1, \lambda) = |Y e^{j\phi_0} - S_1 e^{j\hat{\phi}_1} - S_2 e^{j\hat{\phi}_2}|^2 - \lambda(1 - \cos(\phi_1 - \hat{\phi}_1)), \] (6)

and the resulting LS phase estimate is given by:

\[ \hat{\phi}_1 = \arg \min_{\phi_1} \Lambda(\hat{\phi}_1, \lambda). \] (7)

### 3. Least Squares Phase Estimation

#### 3.1. Closed-form Solution

From Eq. (3), the magnitude of \( Y e^{\phi_0} \) is given by:

\[ Y = \sqrt{S_1^2 + S_2^2 + 2S_1S_2 \cos(\Delta \phi)}, \] (8)

where \( \Delta \phi = \phi_2 - \phi_1 \), is the phase difference. From the above equation, \( \Delta \phi \) is given by:

\[ \Delta \phi = \pm \arccos \left( \frac{Y^2 - S_1^2 - S_2^2}{2S_1S_2} \right), \] (9)

which has ambiguity in sign due to the lack of knowledge regarding the phase-lead or phase-lag relationship between the underlying phase values (see Fig. 2). Therefore, there is always a set of two phase candidates satisfying all the observations in the least squares sense (see also [23] for a detailed derivation of the two sets). Using the phase difference, the mixed signal in the Fourier domain can be written as:

\[ Y e^{j\phi_0} = \left( S_1 + S_2 e^{j\Delta \phi} \right) e^{j\hat{\phi}_1}. \] (10)

Solving Eq. (10) for \( \phi_1 \) we obtain:

\[ \phi_1 = \phi_y + j \log \left( \frac{S_1 + S_2 e^{j\Delta \phi}}{Y} \right). \] (11)

Substituting \( \Delta \phi \) in Eq. (7) and using that the absolute value of \( z^r \in \mathbb{C} \) is given by \( z = \sqrt{z^r z^r} \), with \( z^r \) representing the complex conjugate of \( z \), and that \( 2 \cos(x) = e^{ix} + e^{-ix} \), the Lagrangian function can be rewritten as:

\[ \Lambda(\hat{\phi}_1, \lambda) = Y^2 + S_1^2 + S_2^2 - 2S_1Y \cos(\phi_y - \hat{\phi}_1) \]
\[ - 2S_2Y \cos(\phi_y + \Delta \phi - \hat{\phi}_1) \]
\[ + 2S_1S_2 \Delta \phi - \lambda(1 - \cos(\phi_1 - \hat{\phi}_1)). \] (12)

In order to find the minimum of the Lagrangian function, it is necessary to find \( \hat{\phi}_1 \) and \( \lambda \) at which the gradient of the Lagrangian vanishes, i.e. \( \nabla_{\hat{\phi}_1, \lambda} \Lambda = 0 \). The derivative with respect to \( \hat{\phi}_1 \) is given by:

\[ \frac{\partial \Lambda}{\partial \hat{\phi}_1} = -2S_1Y \left( \sin(\phi_y) \cos(\hat{\phi}_1) - \sin(\hat{\phi}_1) \cos(\phi_y) \right) \]
\[ + 2S_2Y \left( \sin(\phi_y + \Delta \phi) \cos(\hat{\phi}_1) - \sin(\hat{\phi}_1) \cos(\phi_y \Delta \phi) \right) \]
\[ + \lambda \left( \sin(\phi_1) \cos(\hat{\phi}_1) - \sin(\hat{\phi}_1) \cos(\phi_1) \right) = 0, \] (13)

and the derivative with respect to the multiplier \( \lambda \) is:

\[ \frac{\partial \Lambda}{\partial \lambda} = 1 - \cos(\phi_1 - \hat{\phi}_1) = 0. \] (14)

By grouping the terms with \( \sin(\hat{\phi}_1) \) and \( \cos(\hat{\phi}_1) \) from Eq. (13), \( \hat{\phi}_1 \) can be expressed as:

\[ \tan(\hat{\phi}_1) = \frac{A - \lambda B}{C - \lambda D}, \] (15)

with:

\[ A = 2S_1Y \sin(\phi_y) + 2S_2Y \sin(\phi_y + \Delta \phi) \]
\[ B = \sin(\phi_1) \]
\[ C = 2S_1Y \cos(\phi_y) + 2S_2Y \cos(\phi_y + \Delta \phi) \]
\[ D = \cos(\phi_1). \] (16)

Substituting (15) in (14) gives:

\[ \phi_1 - \arctan \left( \frac{A - \lambda B}{C - \lambda D} \right) = 2\pi n, \] (17)

with \( n \in \mathbb{Z} \), i.e., an integer number. Solving for \( \lambda \) results in:

\[ \lambda = \frac{C \tan(\hat{\phi}_1) - A}{D \tan(\hat{\phi}_1) - B}. \] (18)

By substituting Eq. (18) in Eq. (15) we obtain:

\[ \hat{\phi}_1 = \tan(\phi_1). \] (19)

Using Eq. (11) and replacing \( \phi_1 \) in (19), the LSPE is given by:

\[ \phi_1^{ls} = \phi_y + j \log \left( \frac{S_1(\omega) + S_2(\omega) e^{j\Delta \phi}}{Y(\omega)} \right). \] (20)
where \( e^{j\Delta \phi} = \cos(\Delta \phi) + j \sin(\Delta \phi) \) with
\[
\sin(\Delta \phi) = \pm \sqrt{1 - \cos^2(\Delta \phi)}.
\] (21)

3.2. Limit cases

In order for the LSPE to be real valued, the following constraint defined on Eq. (20) should be satisfied
\[
\Re \left( \log \left( \frac{S_1 + S_2 (\cos(\Delta \phi) + j \sin(\Delta \phi))}{Y} \right) \right) = 0,
\] (22)

Using the fact that \( \log(z^*) = \log(z) + j \phi_z \), (22) can be rewritten as
\[
\frac{S_2 \sin(\Delta \phi)}{Y} = \pm \sqrt{1 - \left( \frac{S_1 + S_2 \cos \Delta \phi}{Y} \right)^2}.
\] (23)

By substituting Eq. (9) in the above equation, it can be seen, that this condition is always met, and therefore, \( \phi_{1y}^D \in \mathbb{R} \) for all \( \phi_y, Y, S_1 \) and \( S_2 \).

When \( S_2 \to 0 \) i.e. contribution of the \( s_2 \) to the mixed signal \( y \) is negligible, the LSPE simplifies to
\[
\lim_{S_2 \to 0} \phi_{1y}^D = \phi_y,
\] (24)
which is consistent with [9], and partially explains why phase estimation has been regarded as unimportant in scenarios were the amplitude has a large local SNR. Similar to the discussion in [9], we take into account the fact that a threshold SNR (denoted by \( \text{SNR}_{th} \)) exists above which the noisy phase suffices, and we are motivated to propose the following phase estimator:
\[
\phi_{1}^{\text{LS}}(\omega) = \begin{cases} \phi_{1y}(\omega) & \frac{S_2}{Y}(\omega) < \text{SNR}_{th}, \\ \phi_y(\omega) & \text{otherwise} \end{cases}
\] (25)

The \( \text{SNR}_{th} \) is found from the Monte-Carlo simulation (see Figure 2) found equal to 6 dB which is consistent with previous findings by Vary [9].

4. Experiments

To demonstrate the effectiveness of the proposed LSPE method, in this section we consider two experiments: 1) Monte-Carlo simulation for synthetic signals and 2) real speech signals explained in the following.

4.1. Monte-Carlo Simulation

In the Monte-Carlo simulation, we consider one sinusoid in white Gaussian noise. Both frequency and phase of the sinusoid are taken from a uniform distribution \( \phi \sim U[-\pi, \pi] \) while its amplitude is set to produce a signal-to-noise ratio within the range of -30 to 30 dB. The number of realizations in the Monte-Carlo simulation is set to 200. As the evaluation criterion, we first inspect the MSE phase distortion criterion \( d_{\text{MSE}}(\phi_1, \phi_1) = (\phi_1 - \phi_1)^2 \) for the LS-estimated phase versus the clean phase as the reference. The results are shown in Figure 2 where the performance obtained by the noisy phase and the two sets of LS-estimated phase values are shown and denoted by \( \text{LS} \) and \( \text{LS} \). The two LS-estimates are still ambiguous due to the parity in the sign \( \sin(\Delta \phi) \) given in (21) (see Figure 2). On average, both \( \text{LS} \) and \( \text{LS} \) are similar, and therefore, they seem to overlap. To show the potential of the proposed least-square approach after incorporation of an ambiguity removal method (e.g. the one explained in [23] relying on group-delay constraint), the results obtained after removing the ambiguity in the phase candidates is also shown in Figure 2, denoted by “Best”.

In the MSE sense, the ambiguous LS-estimated phase outperform the noisy phase for \( \text{SNR} \leq -20 \) (dB). In contrast, the ambiguity removed LS-estimates (denoted as “Best”) consistently leads to the best estimates compared to noisy and \( \text{LS} \) or \( \text{LS} \). The noisy phase asymptotically to this best result as the input SNR gets large enough (\( \text{SNR} \geq -5 \) (dB)).

The MSE criterion on phase parameters suffers from wrapping due to phase jumps of \( \pm \pi \) leading to a not necessarily a faithful metric for the evaluation of a phase estimator. This insufficiency of the MSE criterion for quantifying the distortion in phase has been studied in speech coding (see for example [24, 25]). To alleviate this, we also report the performance in terms of \( d_{\text{PESQ}}(\phi_1, \phi_1) \) defined in Eq. (5) which was proposed in [22] for speech enhancement and in [24] for speech coding applications. This makes the metric invariant under module 2\( \pi \) transformation of the estimation error term \( (\phi - \hat{\phi}) \).

From the unwrapped metric, we observe that the noisy phase starts outperforming \( \text{LS} \) and \( \text{LS} \) for input SNRs larger than 6 dB. This confirms the previous observation made in [9] stating that the choice of the noisy phase suffices for local signal-to-noise ratio of larger than 6 dB. However, that observation in [9] was obtained under the prior assumption that the noise probability density function is Gaussian with no restriction on noise pdf. Furthermore, the \( \text{SNR} = 6 \) (dB) reported in [9] was calculated from the maximum phase deviation before reaching the phase perception threshold.

4.2. Proof-Of-Concept Using Phasegrams

As the instantaneous phase spectrum shows a random pattern, in order to demonstrate the effectiveness of the proposed phase estimation method, here we consider group delay as the frequency derivative of phase which excludes the wrapping issue and exhibits a clear harmonic structure following that in the spectrogram. Here, we consider a male utterance corrupted with white noise at \( \text{SNR} = 0 \) dB. Figure 3 shows the spectrogram and group delay plots from left to right, for clean, noisy input, noisy phase reconstructed signal, and reconstructed signal using the proposed LSPE. The PESQ scores obtained by each method is also shown at the top of each panel. The proposed phase estimation...
The proposed phase estimation algorithm can be applied to any single channel speech enhancement method where amplitude and frequency are estimated. To demonstrate the effectiveness of the proposed phase estimation method, we further extend the experiments to realistic speech signals considered as sum of sinusoids. The goal is to justify how much improvement is achievable via replacing the noisy phase with the estimated phase when reconstructing the enhanced speech signal. The amplitude and frequency of the underlying speech signal are assumed to be given\(^5\). Here we choose 100 sinusoids to represent the underlying speech signal at each frame. We select 20 utterances from the TIMIT database [26] corrupted with white and babble noise both selected from NOISEX-92 [27] at global SNR ranging within -20 to 20 dB with 5 dB step. The signals are downsampled to 8 kHz. As our frame setup a Hanning window of length 256 (32 ms) was used with the frameshift of 10 ms. To quantify the obtained improvement in the speech enhancement performance, we consider two metrics: ITU standard in [28] for perceptual evaluation of speech quality (PESQ) and SNRloss [29] for speech intelligibility. Both were shown to have the high correlation with subjective listening results [30]. We note that a lower SNRloss indicates better speech intelligibility.

The results are shown in Figure 4 categorized to white and babble noise scenarios. Significant improvement in both speech quality and speech intelligibility is obtained by the LS-estimated phase followed by ambiguity removal. In terms of PESQ, the LS-estimated phase with ambiguity (denoted by LS1 and LS2), performs on average the same as the noisy phase since they are equal to the noisy phase plus an additive term as a function of \(\Delta \phi\) (as seen in Eq. (20)). In terms of speech intelligibility, the remaining ambiguity in the LS-estimated phase results in information loss, hence degrades the speech intelligibility compared to the input speech signal while after removing the ambiguity in the LS-phase candidates, significant improvement is obtained in intelligibility compared to noisy phase, justified by a lower score of SNRloss. Both observations are consistent with previous observation that the sign of the phase information largely contributes to speech intelligibility [31].

5. Conclusion

In this paper, a least squares (LS) solution is derived for estimating the phase spectrum of an underlying signal corrupted in noise. The proposed solution makes no prior assumption on the probability density of the desired source or interfering noise signals. Throughout Monte-Carlo simulations and realistic speech enhancement experiments, it was demonstrated that the proposed least-squares phase estimator provides lower phase distortion in Monte-Carlo simulations as well as significant improvement speech quality and speech intelligibility compared to when noisy phase spectrum is chosen for signal reconstruction stage in single-channel speech enhancement.

It is a known result that least squares (without any constraint or regularization) is equivalent to maximum likelihood estimation in presence of Gaussian noise. Nevertheless, the proposed LS phase estimator (LSPE) is a more general approach, that do not necessarily relies on the assumption of the probability distribution on the involved signals.

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\(^5\)Future work will report the phase estimation results for a blind scenario where amplitude and frequency are estimated using the conventional speech enhancement methods in the literature [8].
6. References


